FINAL: INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: 21^{st} **May 2021**
The Total points is **105** and the maximum you can score is **100** points.

(1) (15 points) Let $R = k[x]$ be a polynomial ring over a field k and $M = R[y]/(xy)$ be a R-module. For a prime ideals P of R, let M_P denote the localization of M with respect to the multiplicative set $R \setminus P$. Show that M_P is a free R_P-module for all but one prime ideal of R.

(2) (20 points) Let $\mathfrak{m}$ be a maximal ideal of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$. Show that the ring $\mathbb{Q}[x_1, \dots, x_n]/\mathfrak{m}$ is a finite dimensional $\mathbb{Q}$-vector space. Is there an upper bound for the dimension of this vector space? Justify your answer.

(3) (30 points) Let k be an algebraically closed field. Prove or disprove.
  (a) Let $X = Z(xy - 1)$ in $\mathbb{A}^2_k$. There is a surjective morphism of affine varieties from $X \to \mathbb{A}^1_k$.
  (b) Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field k induced from the inclusion of k-algebras $k[Y] \subset k[X]$. The morphism f is surjective.

(4) (20 points) Let X be an algebraic subset of a projective space $\mathbb{P}^3$ over $\mathbb{C}$ defined by the homogeneous polynomial $x_0^2 + x_1^2 + x_2^2 + x_3^2$. Show that X is a variety containing an affine open subset isomorphic to $\mathbb{A}^2$? Prove that X is isomorphic to $\mathbb{P}^2$.

(5) (20 points) Let U be an irreducible curve of degree $d \geq 1$ in $\mathbb{A}^2_{\mathbb{C}}$, i.e. U is the zero set of an irreducible polynomial in two variables of degree d. Let X be its closure in $\mathbb{P}^2$. Let r be the number of points in X not in U. Show that r is between 1 and d. Also show by examples that the two bounds are attained.